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(July 11, 2000)

We analyze the thermodynamical properties of brane-worlds, with a focus on the second model of Randall and Sundrum. We point out that during an inflationary phase on the brane, black holes will tend to be thermally nucleated in the bulk. This leads us to ask the question: Can the black hole - brane-world system evolve towards a configuration of thermal equilibrium? To answer this, we generalize the second Randall-Sundrum scenario to allow for non-static bulk regions on each side of the brane-world. Explicitly, we take the bulk to be a *Vaidya-AdS* metric, which describes the gravitational collapse of a spherically symmetric null dust fluid in Anti-de Sitter spacetime. We calculate the late time behaviour of this system, including 1-loop effects. We argue that at late times a sufficiently large black hole will relax to a point of thermal equilibrium with the brane-world environment. This result has interesting implications for early-universe cosmology.

11.10.Kk, 04.50.+h, 04.65.+e, 11.25.Mj

I. INTRODUCTION: THE THERMODYNAMICS OF INFLATING BRANE-WORLDS

Recently [1], Randall and Sundrum presented a model in which the universe is realized as a $\mathbf{Z}_2$-symmetric positive tension domain wall, or brane, in AdS_5. They analyzed the linearized equation for a graviton propagating in this spacetime. They proved that there is a solution describing a 'bound state', i.e., an integrable wave function corresponding to a massless graviton which is confined to the domain wall. For low energy processes this bound state dominates over the Kaluza-Klein (KK) states, so that Newtonian gravity is recovered as long as the length scale of the AdS space is sufficiently small.

Soon after this model appeared, it was pointed out [2] that the AdS/CFT correspondence gives rise to the Randall-Sundrum model in a certain limit. More precisely, the AdS/CFT correspondence relates the Randall-Sundrum model to an equivalent four dimensional theory consisting of general relativity coupled to a strongly interacting conformal field theory.

Now, the Hawking-Page phase transition [3] manifests itself in the AdS/CFT duality ([4]). Explicitly, there is a critical temperature, T_c, past which thermal radiation in AdS is unstable to the formation of a Schwarzschild black hole. (In fact, for $T>T_c$ there are two values of the black hole mass at which the Hawking radiation can be in equilibrium with the thermal radiation of the background. The lesser of these two masses is a point of unstable equilibrium (it has negative specific heat), whereas the greater mass is a point of stable equilibrium.)

Since the second Randall-Sundrum model may be understood by looking at the AdS/CFT duality in the non-gravity decoupled limit, one would expect that a similar phase transition should occur in that setting. In particular, during an inflationary phase on the brane, the brane-world is a de Sitter hyperboloid embedded in AdS_5, and it will generate an acceleration horizon in the bulk. This horizon will have a temperature, and so we would expect that inflating brane-worlds would be unstable to the creation of bulk (five-dimensional) black holes. In fact, this process was discussed in a recent paper by Garriga and Sasaki [5]. There, the authors studied the 'thermal instantons' which correspond to black holes in AdS, and showed that these instantons describe the thermal nucleation of Schwarzschild-AdS black holes in the presence of a pre-existing $\mathbf{Z}_2$ symmetric inflating brane-world.

For us, this is evidence that we should *assume* that a bulk black hole will be formed during the inflationary epoch of a brane-world universe. It therefore behooves us to answer the question: Can an inflating brane-world and a bulk black hole ever attain thermal equilibrium?

This question is very subtle, because we have to decide what boundary conditions to impose on the bulk fields at the location of the brane-world. Here we assume that the brane-world is a $\mathbf{Z}_2$ symmetric domain wall, so that the brane acts like a reflecting mirror for massless bulk modes. In other words, for us the brane-world is like a uniformly accelerating mirror: A 'black box' with perfectly reflecting, accelerating walls. However, it is well known [6] that a uniformly accelerating mirror in five dimensions (with acceleration parameter A) emits at late times a thermal flux of radiation at a temperature

$$T \sim A, \tag{1}$$

and a corresponding renormalized energy density

$$\langle T_{vv}\rangle \sim A^5, \qquad (2)$$

where v is the null direction normal to the brane. There are thus three basic cases to consider. First, the black hole temperature (T_{BH}) may be greater than the brane-world temperature (T_{BW}), in which case the hole has negative specific heat and it will evaporate in a finite time and have no effect on the late time evolution of the system. Alternatively, the system may be fine-tuned so that $T_{BH} = T_{BW}$ exactly; this would describe thermal equilibrium between the two competing temperatures. Finally, it may be the case that $T_{BW} > T_{BH}$[1]; clearly, this case is of interest because it would appear that the black hole might be unstable to some runaway process where the mass increases indefinitely. We will focus all of our attention on this last case. In order to study this case, we allow the bulk to be non-static, so that radiation emitted by the brane-world will generate a backreation and cause the bulk black hole to grow. Since we are interested in the late time solution, we will only consider ingoing radiation. We work with the signature convention $(-,+,+,+,+)$.

II. GRAVITATIONAL COLLAPSE IN ANTI-DE SITTER SPACE: THE VAIDYA METRIC

In 1951, Vaidya [8] wrote down a metric that represents an imploding (or exploding) null dust fluid with spherical symmetry in asymptotically flat space. Recently [9], this metric has been generalized to describe gravitational collapse in spacetimes with a non-vanishing cosmological constant. Here, we are interested in the metric which describes gravitational collapse in a spacetime which is asymptotically Anti-de Sitter (AdS). This metric is written using 'Eddington-Finkelstein' coordinates, so that it takes the explicit form

$$ds^2 = -e^{2\psi(v,r)} f(v,r)dv^2 + 2\epsilon e^{\psi(v,r)}dvdr + r^2 d\Omega_3^2, \qquad (3)$$

where

$$f(v,r) = k - \frac{2M(v,r)}{r^2},$$

and

$$d\Omega_3^2 = d\chi^2 + R_k^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

with $R_{-1}(\chi) = \sinh(\chi)$, $R_0(\chi) = \chi$ and $R_{+1}(\chi) = \sin(\chi)$ is the metric on hyperbolic space, flat space and the round three-sphere respectively. The function $M(v,r)$, called the *mass function*, is a measure of the total gravitational energy within a radius r. The sign $\epsilon = \pm 1$ indicates whether the null coordinate v is advanced or retarded. If $\epsilon = +1$ then v is advanced time, in which case rays of constant v are ingoing. Likewise, $\epsilon = -1$ means that rays of constant v are outgoing. Since we are interested in collapsing radiation, we will assume $\epsilon = +1$.

In [9] the authors take $\psi(v,r) = 0$, so that the Einstein equations simplify considerably. We also make this assumption, and so we may follow their analysis and conclude that the source for this metric is a 'Type II fluid' [11]. Following [9], it is straightforward to see that if we want to describe gravitational collapse (or the time reverse) in AdS_5 then the appropriate mass function, in general, is

$$M(v,r) = \frac{\Lambda}{12}r^4 + m(v) - \frac{q^2(v)}{r^2}, \qquad (4)$$

where $\Lambda = -(6/l^2)$ is the bulk cosmological constant, and $m(v)$ is an arbitrary function of v which will be determined by the energy density of the radiation in the bulk. $q(v)$ corresponds to the charge of the bulk, if any.

Now that we have a precise idea of what the non-static bulk metric looks like, we turn to the question of how a domain wall moves in such a bulk.

III. THE ISRAEL EQUATIONS OF MOTION: THE COSMOLOGY OF BRANE-WORLDS

The equations of motion for a domain wall, when the effects of gravity are included, are given by the Israel junction conditions. These conditions relate the discontinuity in the extrinsic curvature (K_{AB}) at the wall to the energy momentum (t_{AB}) of fields which live on the wall:

$$[K_{AB} - Kh_{AB}]^{\pm} = \kappa_D^2 t_{AB}. \qquad (5)$$

(see [13] for a derivation of this equation). The gravitational coupling constant, κ_D^2, in arbitrary dimension D, is given by [14]

$$\kappa_D^2 = \frac{2(D-2)\pi^{\frac{1}{2}(D-1)}}{(D-3)(D/2-3/2)!}G_D, \qquad (6)$$

where G_D is the D-dimensional Newton constant. Here, for instance, $\kappa_5^2 = 3\pi^2 G_5$. Given the form of (5), it is obvious that we need to calculate the extrinsic curvature for timelike hypersurfaces which move in the Vaidya-AdS background.

As above, we begin by writing the metric in Eddington-Finkelstein coordinates:

[1] This analysis is similar to that of Yi [7], who studied black holes which are uniformly accelerated by cosmic strings.

$$ds^2 = dv\left[-f(v,r)dv + 2dr\right] + r^2 d\Omega_3^2, \tag{7}$$

where we have assumed that the null coordinate v represents Eddington advanced time. Since we are interested in the cosmological aspects of brane-world evolution, we want the metric induced on the brane-world to assume a manifestly FRW form:

$$ds^2|_{brane-world} = -d\tau^2 + a^2(\tau)\,(t)\,d\Omega_3^2, \tag{8}$$

where the coordinates $x^\mu = (\tau, \chi, \theta, \phi)$ are the coordinates intrinsic to the brane-world. This means that we constrain the timelike hypersurface describing the evolution so that the brane can only shrink or contract as it moves in time, i.e., the position of the brane at a given time should be completely specified by its radial position: $r = r(v) = a(\tau)$. (In what follows we will let $a = a(\tau)$ denote the position of the brane, in order to avoid possible confusion between the coordinate r and the radial position of the brane-world). In this way, the problem is basically reduced to a 1+1 dimensional system. Furthermore, we need only focus on the contributions to the Riemann tensor which are induced by the (r, v)-sector of the metric. Using the fact that τ is the time experienced by observers who move with the brane-world:

$$d\tau = \left(f - 2\frac{da(\tau)}{dv}\right)^{1/2} dv, \tag{9}$$

we may express the problem in the $(\tau, a) - sector$. In particular, if we let '$\dot{a}$' denote differentiation relative to comoving time, i.e., $\dot{a} = da/d\tau$, then one can find the extrinsic curvature through the following relation

$$K_{\mu\nu} = -e^A_\mu e^B_\nu \nabla_A n_B,$$

with

$$e^A_\mu = \left(\frac{1}{f}\left(\dot{a} + \sqrt{\dot{a}^2 + f}\right)\delta^A_v + \dot{a}\delta^A_r\right)\delta_{\mu\tau} + \delta^A_\mu,$$

as the tetrads at the wall and

$$n_A = -\dot{a}\delta^v_A + \frac{1}{f}\left(\dot{a} + \sqrt{\dot{a}^2 + f}\right)\delta^r_A,$$

as the unit normal vector to the hypesurface $a(\tau)$. ∇_A is the covariant derivative associated with the metric (7). The nonvanishing components of extrensic curvature are then

$$\begin{aligned} K^\tau_\tau &= -\frac{1}{2}\frac{2\ddot{a} + f'}{\sqrt{f + \dot{a}^2}} + \frac{1}{2}\frac{\dot{f}}{f^2}\left(\dot{a} + \sqrt{f + \dot{a}^2}\right)^2 \\ &= -\frac{d}{da}\left(\sqrt{f + \dot{a}^2}\right) + \frac{1}{2}\frac{\dot{f}}{f^2}\left(\dot{a} + \sqrt{f + \dot{a}^2}\right)^2 \end{aligned} \tag{10}$$

$$K^\chi_\chi = K^\theta_\theta = K^\phi_\phi = -\sqrt{H^2(\tau) + \frac{f(\tau, a)}{a^2}}\,, \tag{11}$$

with $H \equiv (\dot{a}/a)$.

Given these expressions, we can examine how the non-static bulk is affecting cosmology. That is, we assume that the stress-energy tensor describing the fields which propagate in the brane-world is given as

$$t^A_B = \mathrm{diag}(-(\rho + \rho_\lambda), \mathrm{p} - \rho_\lambda, \mathrm{p} - \rho_\lambda, \mathrm{p} - \rho_\lambda, 0)$$

We emphasize that ρ and p are the energy density and pressure of the ordinary matter, respectively, whereas, ρ_λ is the contribution from the tension of the brane which is simply a Nambo-Goto term. From the Israel equations we may then derive the Friedmann equation on the brane:

$$\begin{aligned} H^2(\tau) = \frac{\Lambda_{eff}}{3} - \frac{k}{a^2} + \left(\frac{8\pi G_4}{3}\right)\rho \\ + \left(\pi^2 G_5\right)^2 \rho^2 + \frac{2m(\tau, a)}{a^4} - \frac{q^2}{a^6}\,, \end{aligned} \tag{12}$$

Henceforth, we shall set $q = 0$ and k the (spatial) curvature of the brane to unity and G_5 to $\sqrt{4G_4/3\pi\rho_\lambda}$. Λ_{eff} is the 4-dimensional cosmological constant on the brane, which is given in terms of the brane tension ρ_λ and the bulk cosmological constant Λ: $\Lambda_{eff} \equiv \Lambda_4 = \left(\frac{\Lambda}{2} + 4\pi G_4 \rho_\lambda\right)$.

Thus, we find that a time-dependent mass in the bulk gives rise to a time-dependent term that scales like radiation. In other words, the collapse of radiation (to form a black hole) in the bulk gives rise to a component of 'Hot Dark Matter' on the brane.

IV. LATE TIME BEHAVIOUR OF THE SYSTEM

We have seen that the thermal nucleation of a black hole in the bulk gives rise to a time-dependent radiation term on the brane. To understand how this term will affect brane-world cosmology, we would like to solve for the back-reaction generated by 1-loop effects in the bulk. As this is a rather difficult problem, we will simply estimate the late-time behaviour of the system.

Now, there could be many components of stress-energy in the bulk, which could contribute to the mass of the black hole. Here, we assume that the only matter in the bulk is radiation which is emitted by the brane-world as it accelerates away from the bulk black hole. In other words, we want to see if it is even *consistent* to ignore the Hawking radiation. Following [9], we may then relate the intensity of the stress-energy tensor in the bulk (equation (2)) with the rate of change in the mass function:

$$A^5 \sim \langle T_{vv}\rangle = \frac{2\dot{m}(\tau, a)}{\kappa_5^2 {r_h}^2}\frac{f}{\dot{a} + \sqrt{\dot{a}^2 + f}} \tag{13}$$

where r_h denotes the horizon radius and we have used equation (9). During inflation, we expect that the equation of state on the brane-world is pure vacuum, so that $\rho = -p$, and everything will be dominated by Λ_{eff}. Therefore we set $\rho = 0$ in equation (11).

The acceleration five-vector for a domain wall moving in a spherically symmetric background is given as [13]:

$$A^A = -K_{\tau\tau} n^A \tag{14}$$

In other words, the *magnitude* of the acceleration is just $A = -K_{\tau\tau}$, and so the equation for the mass function becomes

$$\frac{2\dot{m}(\tau, a)}{\kappa^2 {r_h}^2} \sim -\frac{1}{f}\left[\frac{d}{da}\left(\sqrt{f+\dot{a}^2}\right) - \frac{1}{2}\frac{\dot{f}}{f^2}\left(\dot{a}+\sqrt{f+\dot{a}^2}\right)^2\right]^5 \left(\dot{a}+\sqrt{\dot{a}^2+f}\right) \tag{15}$$

We now turn to the late time evolution of this equation. Before the black hole forms, the brane is exponentially expanding. Since we *assume* the brane doesn't fall into the black hole once the hole forms, the brane should still be exponentially expanding once gravitational collapse begins. Consequently, suppose $a \sim e^t$ at late times. Then it is straightforward to show that $m \sim e^{-2t}$, and so it would seem that the black hole shrinks. In fact, this means that at late times our approximation always breaks down: At late times we will have to include the effects of Hawking radiation. Eventually, the black hole will emit a temperature comparable to the temperature of the brane-world, and the system will equilibrate. From the point of view of the AdS/CFT duality, this makes sense. Truncating AdS_5 with a de Sitter brane introduces a definite temperature into the system, namely, the temperature of the 'mirror'. As long as sufficiently large bulk black holes are allowed, we know that these holes can be in thermal equilibrium with thermal radiation. The main shortcoming of this analysis is that we assumed reflecting boundary conditions for all bulk modes at the brane-world. Presumably, there may exist fields on the brane which can be excited by bulk gravitons (for example). This would mean that the boundary conditions at the brane would have to allow for some absorption. We will consider such boundary conditions in an upcoming paper [16].

To summarize: We have shown that the Friedmann equations of the second Randall-Sundrum model will generically contain a hot dark radiation term which has an intensity that is variable but will settle down to a fixed value during inflation. Once inflation ends, the mass function will presumably change in some way; it remains to be shown that the system will still equilibrate once the equation of state on the brane is no longer pure vacuum. It is amusing to note that this varying mass function is reminiscent of the 'C-field' introduced by Hoyle and Narlikar in their steady state model and later on in the quasi-steady state model of the universe (see e.g. [15] and references therein). Both our dynamical mass function and their C-field can pump energy into the universe, so that total energy on the brane is not conserved. We will have more to say about the cosmology of this model in an upcoming paper [16].

Acknowledgements

We thank R. Battye, E. Bertschinger, A. Guth, P. Khorsand, P. Kraus, P. Mannheim, L. Randall and C. Vafa for useful conversations. AC and AK are supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818. A.N. is supported by NSF grant ACI-9619019. CTP preprint # 2998.

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